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A new minimum volume straight cooling fin taking into account the "length of arc"

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Abstract

The problem of determining the shape of a straight cooling fin of minimum volume without the "length of arc" assumption is addressed. Proceeding from the conventional assumptions of one-dimensionality of the temperature distribution and its linearity for the minimum volume fin we found the profile of the optimum fin to be a circular arc and computed its geometric parameters. The volume of the optimum circular fin found in this paper is 6.21–8 times smaller than the volume of the corresponding Schmidt's parabolic optimum fin. The optimum circular fin tends to be shorter and to have a larger base height than Schmidt's fin.

1. Introduction

Fins are surface extensions frequently used in heat exchange devices for the purpose of increasing heat transfer rates between solid surfaces at a high temperature and surrounding fluids at a low temperature, see e.g. [1]. Numerous theoretical and experimental studies [2–7] suggest that the thermal performance of fins depends critically on their shape. Fins with variable thickness were first considered in the classical works by Harper and Brown [2] and Schmidt [3]. Owing to the combined action of axial conduction and transverse convection, the temperature in fins varies mostly longitudinally so that the transversal temperature change is generally viewed as insignificant. This assumption is commonly referred to as "one-dimensionality of the temperature distribution." As usual, the analysis of the thermal performance of straight fins will be limited to the case of symmetric fins (see Fig. 1, where XZ is the plane of symmetry of the fin).

Maximization of heat transfer from a heated surface through a straight fin to a surrounding fluid for a given amount of material was first analyzed in 1926 by Schmidt [3]. Equivalently, the problem consists of minimizing the volume of a straight fin that dissipates a given amount of heat to the ambient fluid. Invoking an illuminating intuitive argument, Schmidt found that the most favorable dimensions for a straight fin are those that produce a linear temperature profile along the fin. In this case the heat flux along the fin is uniform. Three decades later, Duffin [8] confirmed the validity of Schmidt's analysis employing rigorous methods of calculus of variations.

Proceeding from the linear temperature distribution along the fin length, Schmidt found that the optimum profile of a straight fin is convex parabolic. The shape of the parabola is completely determined by two prespecified quantities of thermal nature: the ratio $\gamma = h/k$ of the heat transfer coefficient h and the thermal conductivity k of the fin material and the dimensionless quantity $\rho = q_0/(k\theta_0)$, where q_0 is the heat flow through the fin semi-base per unit depth and θ_0 is the difference between the temperatures of the heated surface and the surrounding fluid, see formulas (11) below. For a detailed treatment of Schmidt's work the reader is referred to [4].

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Nomenclature

A	fin profile semi-area
h	heat transfer coefficient
k	thermal conductivity
L	fin length
q_0	heat transfer rate at the fin semi-base per unit depth
W	fin depth
x	longitudinal space variable
y	transversal space variable
y_0	fin semi-thickness at the base

Greek symbols

γ	h/k ratio
θ	temperature excess
θ_0	temperature excess at the fin base
ρ	dimensionless q_0, $q_0/(k\theta_0)$

Superscript

*	refers to the optimum fin found in the present work

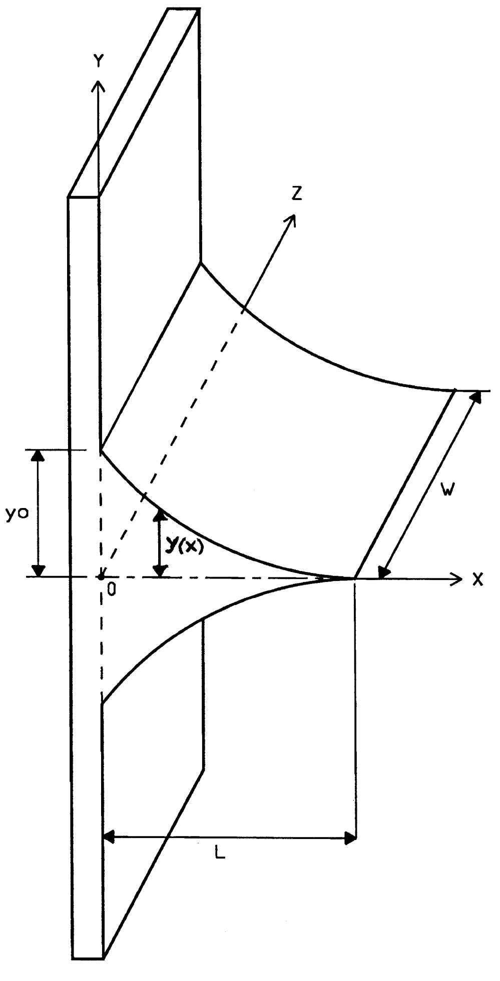

Fig. 1. Straight cooling fin.

A key assumption shared by the three classical publications [2–4] and many subsequent works for that matter (see e.g. [5,6,8]) is the omission of the "length of arc" of the fin profile. From a physical standpoint, the "length of arc" condition is equivalent to the assumption that heat is dissipated from the fin to the surrounding fluid in the direction orthogonal to the plane of symmetry of the fin (that is, along Y axis in Fig. 1). In reality, however, the direction of heat flux from the fin to the fluid is orthogonal to the fin surface.

Maday [7] was the first researcher who addressed the impact that the "length of arc" assumption exerts on the optimum profile shape of straight fins. He pointed out that the differential area element of the semi-surface per unit depth applicable to the straight fin should be expressed by the relation $dS = \sqrt{1 + (y')^2}\,dx$, where $y = y(x)$ stands for the fin profile function. In particular, the approximation $dS \simeq dx$ is equivalent to the elimination of the "length of arc."

The problem of optimum fin design was set up in [3,4] as a search for the optimum fin shape, length L, and semi-height at the fin base y_0, given the thermal parameters q_0, θ_0, h and k. In contrast to this, Maday [7] was looking for a minimum volume fin with a *fixed* y_0. He formulated the problem as a two-point boundary value problem, and solved it numerically using the Pontryagin's Maximum Principle. The optimum profile reported in [7] is reasonably close to Schmidt's convex parabolic profile for a large initial portion of the fin length, but closer to the end contains some wavy irregularities. The volume of the fin found in [7] was only slightly smaller than the volume of Schmidt's fin with the same height. For example, for $\rho = 0.1$, the wavy optimal shape is about 15% shorter than Schmidt's fin and produces a 1.6% volume reduction over the latter, while for $\rho = 0.02$, their lengths are close and the volume reduction is 1%. It was also claimed in [7] that the waviness diminishes when ρ tends to zero. An important numerical finding in [7] is that, with the "length of arc" assumption lifted, the temperature distribution for the optimum fin is still linear.

The goal of the present work is to obtain an *exact analytic form* for the optimum profile of a straight fin that minimizes the fin volume and procures dissipation of a given heat flow per unit depth to the surrounding fluid. When solving this problem, we consider the thermal parameters q_0, θ_0, h and k as fixed while all geometric parameters of the fin (semi-height at the fin base, length, and profile shape) are subject to optimization. In our mathematical development we eliminate the "length of arc" assumption altogether.

The optimum fin profile obtained in this work turns out to be a circular arc. It tends to be shorter and to have a larger semi-height at the base than the corresponding Schmidt's fin. The most striking finding is that the volume of the optimum circular fin is 6.21–8 times smaller than the volume of Schmidt's fin.

Although the problem of optimum fin design is formulated and solved for *given* heat flow q_0 and temperature excess θ_0, the actual heat flow generated by the optimum fin, as determined by the temperature distribution, will be somewhat different from q_0, due to various simplifying assumptions involved in the derivation of the optimum fin shape. To account for this departure, to gauge the thermal efficiency of the fin for other values of heat flow and temperature excess at the fin base (for which the fin is not necessarily optimal), and to allow for comparison of various fin designs, the fin *performance* ε is used as a figure-of-merit. The latter is usually defined as the ratio of the heat transfer rate through the fin base to the heat transfer rate without the fin. Thus, for straight symmetric fins, $\varepsilon = q_0/(hy_0\theta_0)$. This quantity can be determined either numerically through a more advanced mathematical model or experimentally. It was shown by Graff and Snider [9] that omission of the "length of arc" always *underestimates* the heat transfer performance of a straight fin. Therefore, the advantage of the optimum circular fin over Schmidt's fin (whose design neglected the curvature of the fin boundary) in terms of thermal performance may appear to be less staggering than in terms of the fin volumes. For a detailed comparison of the optimum circular fin with Schmidt's fin, the reader is referred to Section 3.

Similar to the findings of the works [3,4], the geometry of the optimum circular fin profile is completely determined by the parameters γ and ρ. This explains why the ratios of the volume of the circular optimum fin to the volume of Schmidt's fin found in the present work are significantly smaller than those reported in [7]: fixing the fin semi-height at the base y_0 forces one to deal with suboptimal fins.

2. Problem formulation

We are looking for a straight homogeneous symmetric fin (see Fig. 1) with minimum volume that dissipates a given amount of heat $2q_0$ per unit depth (length in the Z direction) from a heated wall at a given temperature to the ambient fluid. The fin geometry is determined by the semi-profile function $y(x)$, $0 \leqslant x \leqslant L$, where L is the fin length. Let $y_0 := y(0)$ be the semi-height of the fin base. Denote by θ the difference between the temperature at some point in the fin and the temperature of the ambient fluid, and let $\theta_0 > 0$ be the corresponding temperature excess at the wall.

We proceed from the following assumptions:

1. The thermal conductivity k of the fin and the heat transfer coefficient h are temperature-independent.
2. The temperature excess θ in every vertical fin cross-section is constant: $\theta = \theta(x)$. Then the same is true for the heat flow $2q$ through the fin cross-section per unit depth.
3. The processes of heat conduction along the fin and heat convection from the fin to the fluid are steady-state.
4. Heat loss through the two extreme fin cross-sections (at $z = 0$ and $z = W$) is negligible. Also, heat dissipation by radiation is insignificant. Thus, our analysis is two-dimensional.
5. Function y is continuous on $[0, L]$ and differentiable on $(0, L)$.

The equations

$$q = -ky\frac{\mathrm{d}\theta}{\mathrm{d}x} \tag{1}$$

and

$$\frac{\mathrm{d}q}{\mathrm{d}x} = -h\theta\sqrt{1 + (y')^2} \tag{2}$$

govern the heat transfer along the fin and from the fin to the ambient fluid, respectively. These equations satisfy the following boundary conditions:

$$\theta(0) = \theta_0, \tag{3}$$

$$q(0) = q_0 \tag{4}$$

and

$$y(L)\frac{\mathrm{d}\theta}{\mathrm{d}x}(L) = 0. \tag{5}$$

Eq. (5) reflects the fact that the heat flow at the tip of the fin is equal to 0 which follows from the assumption that the entire heat flux from the wall is dissipated by the fin. The latter assumption can also be expressed in the form

$$h\int_0^L \theta\sqrt{1 + (y')^2}\mathrm{d}x = q_0. \tag{6}$$

It is clear that given (1), (2) and (4), Eqs. (5) and (6) are equivalent. Let also

$$A = \int_0^L y(x)\,\mathrm{d}x \tag{7}$$

be the fin cross-sectional semi-area to be minimized.

We now give a mathematical formulation of the problem:

Given the temperature excess θ_0 and heat flow per unit depth q_0, find the fin length L, the semi-height at the fin base y_0, and the fin shape $y(x)$, $0 \leqslant x \leqslant L$, with $y(0) = y_0$, for which Eqs. (1) *and* (2) *together with the boundary conditions* (3)–(5) *are satisfied, and the area functional* (7) *is minimized.*

Following a heuristic argument by Schmidt [3], we will assume that for the optimum fin configuration, the temperature profile along the fin is linear:

$$\theta(x) = \theta_0 - mx, \quad 0 \leqslant x \leqslant L, \tag{8}$$

where $m > 0$. Eq. (1) suggests that the linearity of the temperature profile is equivalent to the heat flow per unit cross-sectional area being constant.

Under the "length of arc" assumption Eq. (2) takes on a simpler form

$$\frac{\mathrm{d}q}{\mathrm{d}x} = -h\theta.$$

Significant simplification of Eq. (2) is the reason why the "length of arc" assumption is widely used in the literature related to optimum fin design. It should be mentioned, however, that Schmidt's argument, that leads to the linearity of the temperature profile for the optimum fin, is independent of the "length of arc" assumption (see [7] for a numerical corroboration).

The geometric characteristics and shape of the optimum parabolic fin found by Schmidt [3] can be expressed through the parameter

$$\gamma = \frac{h}{k} \tag{9}$$

and the dimensionless quantity

$$\rho = \frac{q_0}{k\theta_0}. \tag{10}$$

Specifically,

$$L = \frac{2\rho}{\gamma}; \quad y_0 = \frac{2\rho^2}{\gamma}; \quad A = \frac{4\rho^3}{3\gamma^2};$$
$$y(x) = \frac{\gamma}{2}\left(\frac{2\rho}{\gamma} - x\right)^2, \quad 0 \leqslant x \leqslant L. \tag{11}$$

Note that $y(L) = 0$. Additionally, for Schmidt's optimum fin one has

$$m = \frac{\theta_0}{L} = \frac{\gamma\theta_0}{2\rho}$$

which in view of (8) implies that $\theta(L) = 0$.

A straightforward computation shows that the length of the parabolic fin profile (11) is

$$S = \frac{1}{2\gamma}\left[2\rho\sqrt{1+4\rho^2} + \ln(2\rho + \sqrt{1+4\rho^2})\right]. \tag{12}$$

3. Solution of the problem

Differentiating Eq. (1) in x with Eq. (2) and notation (9) taken into account we have

$$y\theta'' + y'\theta' - \gamma\theta\sqrt{1+(y')^2} = 0.$$

For the linear temperature function (8), this equation becomes

$$y' = -\gamma(\mu - x)\sqrt{1+(y')^2}, \tag{13}$$

where

$$\mu := \frac{\theta_0}{m} > 0. \tag{14}$$

It follows from the second law of thermodynamics that $\theta(x) \geqslant 0$, $0 \leqslant x \leqslant L$, which implies that $\theta_0 \geqslant mL$ or equivalently

$$L \leqslant \mu. \tag{15}$$

Hence in view of (13) we have $y'(x) \leqslant 0$, $0 \leqslant x \leqslant L$. Then Eq. (13) yields

$$y'(x) = -\frac{\gamma(\mu - x)}{\sqrt{1-\gamma^2(\mu-x)^2}}, \quad 0 \leqslant x \leqslant L, \tag{16}$$

where we must have $\gamma\mu \leqslant 1$, that is,

$$\mu \leqslant \frac{1}{\gamma}. \tag{17}$$

Observe that in view of $\theta'(L) = m > 0$ the boundary condition (5) becomes simply

$$y(L) = 0. \tag{18}$$

The solution of Eq. (16) satisfying the boundary condition (18) is given by

$$y(x) = \frac{1}{\gamma}\left[\sqrt{1-\gamma^2(\mu-L)^2} - \sqrt{1-\gamma^2(\mu-x)^2}\right]. \tag{19}$$

Therefore,

$$y_0 = \frac{1}{\gamma}\left[\sqrt{1-\gamma^2(\mu-L)^2} - \sqrt{1-\gamma^2\mu^2}\right]. \tag{20}$$

Setting $x = 0$ in (1) we have $q_0 = ky_0m$, whence using (10), (14) and (20) we find that

$$\gamma\rho\mu = \sqrt{1-\gamma^2(\mu-L)^2} - \sqrt{1-\gamma^2\mu^2}. \tag{21}$$

Eq. (19) describes an arc of the circle

$$(x-\mu)^2 + (y-v)^2 = \frac{1}{\gamma^2}, \quad \text{where} \quad v = \frac{1}{\gamma}\sqrt{1-\gamma^2(\mu-L)^2}. \tag{22}$$

Thus, under the assumptions formulated above, *every* fin shape that produces a linear temperature distribution is necessarily circular. The family of circles (22) depends on two parameters μ, $L > 0$ which are subject to the conditions

$$0 < L \leqslant \mu \leqslant \frac{1}{\gamma}, \tag{23}$$

see (15) and (17). Eq. (21) restricts it to a one-parametric subfamily. In what follows, we optimize the remaining parameter to obtain a fin with the smallest cross-sectional area.

Integrating the function (19) and invoking (7) we find that

$$2\gamma A = (L+\mu)\sqrt{1-\gamma^2(\mu-L)^2} - \mu\sqrt{1-\gamma^2\mu^2} - \frac{1}{\gamma}\left(\sin^{-1}(\gamma\mu) - \sin^{-1}[\gamma(\mu-L)]\right). \tag{24}$$

Therefore, the problem is to minimize the right-hand side of (24) over all L, $\mu > 0$ that satisfy Eq. (21) and the constraints (23). To solve this problem, denote

$$\alpha := \sin^{-1}(\gamma\mu) \quad \text{and} \quad \beta := \sin^{-1}[\gamma(\mu-L)], \tag{25}$$

and observe that the constraints (23) translate into the inequalities $0 \leqslant \beta < \alpha \leqslant \pi/2$. We solve Eq. (25) for μ and L to obtain

$$\mu = \frac{\sin\alpha}{\gamma} \quad \text{and} \quad L = \frac{1}{\gamma}(\sin\alpha - \sin\beta). \tag{26}$$

Combining Eqs. (21), (24) and (26) we conclude that in terms of α and β our problem consists of minimizing the function

$$2\gamma^2 A = (\sin\alpha - \sin\beta)\cos\beta + (\cos\beta - \cos\alpha)\sin\alpha - \alpha + \beta \tag{27}$$

for α, β subject to the conditions

$$\rho\sin\alpha + \cos\alpha = \cos\beta, \quad 0 \leqslant \beta < \alpha \leqslant \frac{\pi}{2}. \tag{28}$$

Introduce the function

$$\psi(\alpha) := \rho\sin\alpha + \cos\alpha = \sqrt{1+\rho^2}\cos(\alpha-\omega), \quad \text{where } \omega = \tan^{-1}\rho.$$

Observe that $\psi(0) = \psi(2\omega) = 1$ and moreover, $\psi(\alpha) > 1$ for $\alpha \in (0, 2\omega)$. In particular, for $\rho \geqslant 1$, we have $\omega \geqslant \pi/4$ so that Eq. (28) has no solution. Physically, this means that in the case $\rho \geqslant 1$ the circular fin in question cannot dissipate the entire heat flux $2q_0$, given the base temperature excess θ_0 and the thermal conductivity k.

From now on we will be assuming that $0 < \rho < 1$. In this case, for each $\alpha \in [2\omega, \pi/2]$, there exists a unique angle β satisfying conditions (28). Then, the right-hand side of Eq. (27) takes on the form

$$\phi(\alpha) = \left[\sin\alpha - \sqrt{1-(\rho\sin\alpha+\cos\alpha)^2}\right](\rho\sin\alpha + \cos\alpha) + \rho\sin^2\alpha + \cos^{-1}(\rho\sin\alpha+\cos\alpha) - \alpha.$$

Thus, our problem reduces finally to minimizing the function $\phi(\alpha)$ for $2\omega \leqslant \alpha \leqslant \pi/2$. Graphs of the function ϕ for various values of ρ are shown in Fig. 2. The graphs suggest that the minimum value of the function ϕ on the interval $[2\omega, \pi/2]$ is attained for $\alpha = \pi/2$ and equals

$$\Phi(\rho) := 2\rho - \rho\sqrt{1-\rho^2} - \sin^{-1}\rho, \tag{29}$$

while for the corresponding value of β we have $\beta = \cos^{-1}\rho$. From Eqs. (14), (20), (26) and (27) we find the following quantities that characterize the optimum circular fin:

$$\mu^* = 1/\gamma, \quad m^* = \gamma\theta_0, \quad L^* = \frac{1}{\gamma}(1-\sqrt{1-\rho^2}),$$
$$y_0^* = \frac{q_0}{h\theta_0} = \frac{\rho}{\gamma}, \quad A^* = \frac{\Phi(\rho)}{2\gamma^2}. \tag{30}$$

Observe that, by contrast to Schmidt's fin, the temperature excess at the tip of the optimum circular fin is nonzero. In fact, it follows from Eqs. (8) and (30) that

$$\theta(L^*) = \theta_0 - m^*L^* = \theta_0\sqrt{1-\rho^2}.$$

Expanding the functions involved in (29) we find that, for small ρ,

$$L^* = \frac{\rho^2}{2\gamma} + \mathrm{O}(\rho^4), \quad \Phi(\rho) = \frac{\rho^3}{3} + \mathrm{O}(\rho^5) \quad \text{and} \quad A^* = \frac{\rho^3}{6\gamma^2} + \mathrm{O}(\rho^5), \tag{31}$$

compare with (11).

It follows from (19) and (30) that the optimum circular fin profile is given by the function

$$f^*(x) = \frac{1}{\gamma}\left[\rho - \sqrt{1-(1-\gamma x)^2}\right], \quad 0 \leqslant x \leqslant \frac{1}{\gamma}(1-\sqrt{1-\rho^2}). \tag{32}$$

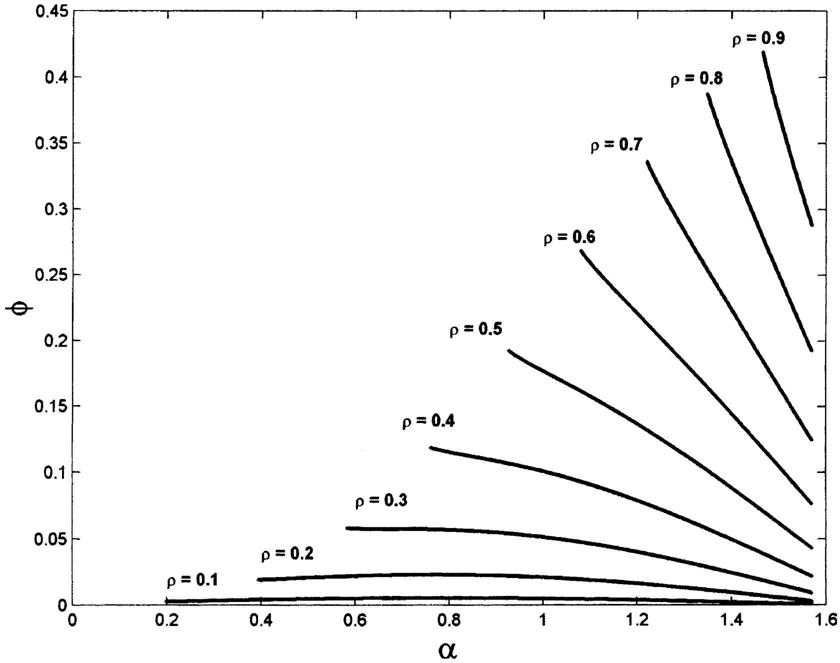

Fig. 2. Graphs of the function $\phi(\alpha)$ $\rho = 0.1, \ldots, 0.9$.

The graph of this function represents an arc of the circle (see Fig. 3) given by the equation

$$\left(x - \frac{1}{\gamma}\right)^2 + \left(y - \frac{\rho}{\gamma}\right)^2 = \frac{1}{\gamma^2},$$

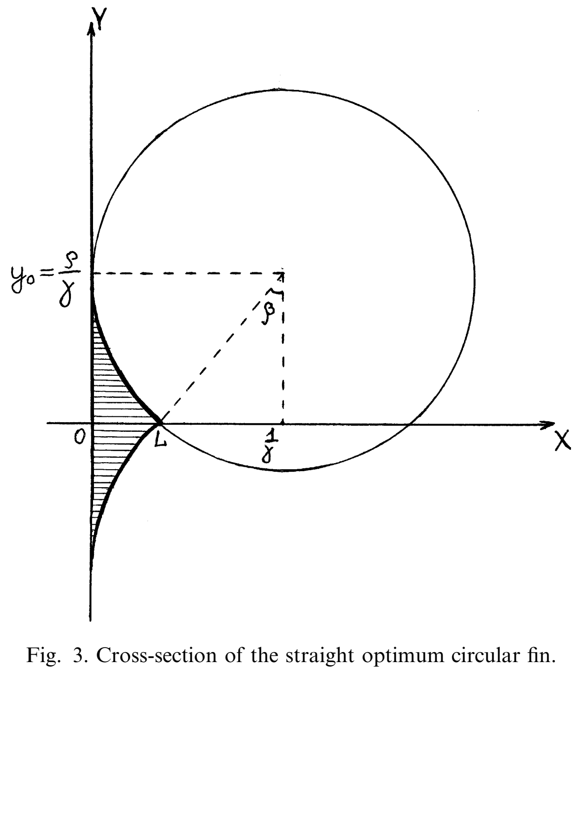

Fig. 3. Cross-section of the straight optimum circular fin.

compare with (22). Clearly, the central angle of the arc is $\pi/2 - \beta = \sin^{-1}\rho$, and hence for the length of the arc we have

$$S^* = \frac{\sin^{-1}\rho}{\gamma}. \tag{33}$$

4. Comparison with Schmidt's fin

Comparing the geometric parameters of the optimum circular fin (30) and (32) with those of Schmidt's fin given in (11) we conclude that, for the same values of the thermal quantities γ, ρ and same depth, the optimum circular fin is shorter and has a larger base. The saving in the amount of material required for manufacturing the optimal circular fin, as compared to the Schmidt's fin, is equal to the ratio A/A^* of their semi-profile areas that can be computed from Eqs. (30) and (11) as follows:

$$g(\rho) := \frac{A}{A^*} = \frac{8\rho^3}{3\Phi(\rho)} = \frac{8\rho^3}{3(2\rho - \rho\sqrt{1-\rho^2} - \sin^{-1}\rho)}. \tag{34}$$

The graph of the function $g(\rho)$ for $0 \leqslant \rho \leqslant 1$ displayed in Fig. 4 suggests that the function $g(\rho)$ is decreasing and takes values between $g(1) = 16/[3(4-\pi)] \simeq 6.21$ and $g(0) = 8$, see the expansion for $\Phi(\rho)$ in (31). Thus, the

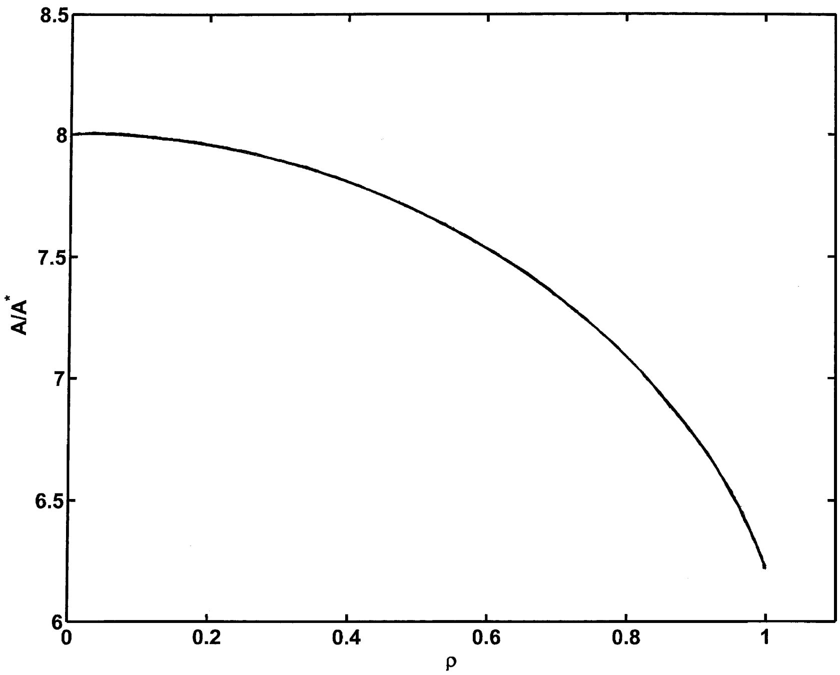

Fig. 4. The ratio A/A^* of the cross-sectional semi-areas of Schmidt's fin and the optimum circular fin as a function of parameter ρ.

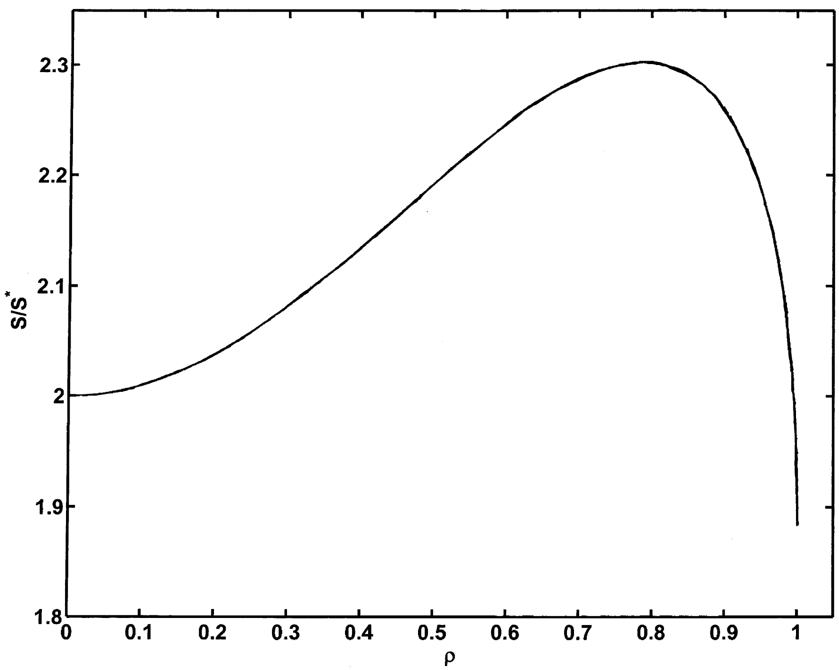

Fig. 5. The ratio S/S^* of the semi-profile arc lengths of Schmidt's fin and the optimum circular fin as a function of parameter ρ.

optimum circular fin requires from 6.21 to 8 times less material than Schmidt's optimum fin.

Another aspect related to the advantage of the optimum circular fin over Schmidt's fin is revealed when one compares the amount of heat flow dissipated by the two fins made from the same material and having the same depth and volume. Assuming that the parameters h, k, and θ_0 for both fins are equal, we denote by q_0^* and q_0 the heat transfer rates per unit depth for which the optimum circular and Schmidt's fins, respectively, were designed, and by ρ^* and ρ the corresponding dimensionless quantities defined in Eq. (10). Using Eqs. (11) and (30) we derive from $A^* = A$ that

$$\frac{\Phi(\rho^*)}{2\gamma^2} = \frac{4\rho^3}{3\gamma^2},$$

hence it follows from (34) that

$$g(\rho^*) = \left(\frac{\rho^*}{\rho}\right)^3.$$

Therefore,

$$\frac{q_0^*}{q_0} = \frac{\rho^*}{\rho} = g(\rho^*)^{1/3}.$$

This formula suggests that the optimum circular fin dissipates from 1.74 to 2 times larger heat flow than the corresponding Schmidt's fin.

The advantage in thermal performance of the optimum circular fin, as compared to Schmidt's fin, is due to the fact that the distance from the heated wall to the fin surface is much shorter for the former than for the latter. Another factor that has a bearing on the thermal performance of a fin is its surface area. For two straight symmetric fins with the same depth and profiles vanishing at the end, the ratio of their surface areas is equal to the ratio of their semi-profile arc lengths. From Eqs. (12) and (33) we find that

$$v(\rho) := \frac{S}{S^*} = \frac{2\rho\sqrt{1+4\rho^2} + \ln(2\rho + \sqrt{1+4\rho^2})}{2\sin^{-1}(\rho)}.$$

The graph of the function $v(\rho)$ for $0 \leqslant \rho \leqslant 1$ shown in Fig. 5 suggests that, although Schmidt's fin is substantially longer and has more than 6 times larger profile area than the optimum circular fin, its surface area is only 1.88 to 2.30 times larger than that of the circular fin.

There remains an important question as to what extent the heat flow in the optimum circular fin (as well as in Schmidt's fin) satisfies the one-dimensionality assumption. Answering this question, as well as conducting a comprehensive comparative analysis of the thermal performance of these fins, calls for an experimental study.

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